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Results on the edge double Roman domination number of a graph

Mina Valinavaz[a,*]

[a]*Department of Mathematics, Azarbaijan Shahid Madani University Tabriz, I.R. Iran.*

Abstract

An *edge double Roman dominating function* (EDRDF) on a graph G is a function $f : E(G) \to \{0,1,2,3\}$ satisfying the condition that such that every edge e with $f(e) = 0$, is adjacent to at least two edge e, e' for which $f(e) = f(e') = 2$ or one edge e'' with $f(e'') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The Edge double Roman dominating number of G, denoted by $\gamma'_{dR}(G)$, is the minimum weight $w(f) = \sum_{e \in E(G)} f(e)$ of an edge double Roman dominating function f of G. In this paper, we introduction some results on the edge double Roman domination number of a graph.

Keywords: Double Roman dominating function, Double Roman domination number, Edge double Roman dominating function, Edge double Roman domination number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$.

An *Edge Roman dominating function*(ERDF) of graph G is a function $f : E(G) \longrightarrow \{0,1,2\}$ satisfying the condition that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$. The *Edge Roman domination number* of a graph G, denoted by $\gamma'_R(G)$, is the minimum weight $w(f) = \sum_{e \in E(G)} f(e)$ of an Edge Roman dominating function of G. The concept of edge Roman domination has been several variants of domination, see for example [9, 10, 14, 15, 8, 4]

A *Edge double Roman dominating function*(EDRDF) of graph G is a function $f : E(G) \longrightarrow \{0,1,2,3\}$ having the property that if $f(e) = 0$, then edge e has at least two neighbors assigned 2 under f or one neighbor e' with $f(e') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The *weight* of an edge double Roman dominating number of f, denote by $\omega(f)$, is the value $\sum_{e \in E(G)} f(e)$. The weight of a EDRDF, $\sum_{e \in E(G)} f(e)$. The minimum weight of a EDRDF is the edge double roman domination number of G, denoted by $\gamma'_{dR}(G)$. If f is a EDRDF in a graph G, then we simply can represent f by $f = (E_0, E_1, E_2, E_3)$(or $f = (E_0^f, E_1^f, E_2^f, E_3^f)$ to refer to f), where $E_0 = \{e \in E(G) : f(e) = 0\}, E_1 = \{e \in E(G) : f(e) = 1\}, E_2 = \{e \in E(G) : f(e) = 2\}$, and $E_3 = \{e \in E(G) : f(e) = 3\}$.

*Corresponding author

Email address: `first-author@example.com` (Mina Valinavaz)

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In this note we initiate the study of the Edge double Roman domination in graphs and present some (sharp) bounds for this parameter. In addition, we determine the Edge double Roman domination number of some classes of graphs.

2. Graphs with Small or large Edge double Roman Domination Number

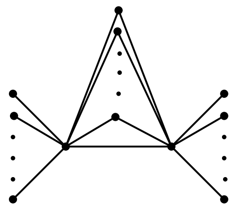

Figure 1: Structure of graphs in the family $\mathcal{T}$

Let $\mathcal{T}$ be the class of all graphs G such that $G = K_2 \vee \overline{K_{n-2}}$ or G is obtained from $K_2 \vee \overline{K_{n-2}}$ by removing at most one edge incident with x for every vertex $x \in V(K_2 \vee \overline{K_{n-2}})$, where $V(K_2) = \{u, v\}$.

Proposition 2.1. *Let* G *be a connected graph of size* $m \geqslant 2$. *Then* $\gamma'_{dR}(G) = 3$ *if and only if* $G \in \{K_4 - e\} \cup \mathcal{T}$, *where* $e \in E(K_4)$.

Proof. Assume that $\gamma'_{dR}(G) = 3$. Let $f = (E_0^f, E_1^f, E_2^f, E_3^f)$ be a $\gamma'_{dR}(G)$-function of G such that $E_1 = \emptyset$(by proposition **??**) and $|E_3^f|$ is maximum. Assume that $|E_3^f| = 1$. Let $E_3^f = \{xy\}$. Clearly, $\{xy\}$ is an edge dominating set of G. Then, each edge must be incident on x or y. Thus, $G = K_2 \vee \overline{K_{n-2}}$ or is obtained from $K_2 \vee \overline{K_{n-2}}$ by removing at most one edge incident with u for every vertex $u \in V(G) - \{x, y\}$. Consequently, $G \in \{K_4 - e\} \cup \mathcal{T}$. The converse is obvious. □

Proposition 2.2. *Let* G *be a connected graph of size* $m \geqslant 4$. *Then* $\gamma'_{dR}(G) = 4$ *if and only if* $G \in \{K_4, C_4\}$.

Proof. Assume that $\gamma'_{dR}(G) = 4$. Let $f = (E_0^f, E_1^f, E_2^f, E_3^f)$ be a $\gamma'_{dR}(G)$-function of G such that $E_1 = \emptyset$(by proposition **??**) and $|E_2^f|$ is maximum. Assume that $|E_2^f| = 2$. Let $E_2^f = \{uv\}, \{u'v'\}$. by definition $\gamma'_{dR}(G)$, each other edge must be incident on u or$v$ and u' or v'. It is easy to see that $G = K_4$ and $G = C_4$. The converse is obvious. □

Let $\mathcal{G}$ be the class of all graphs H such that H is obtained from a graph $G \in \mathcal{T}$ by adding an edge between the vertices $V(G) - \{u, v\}$ or adding a leaf to a vertex of $V(G) - \{u, v\}$.

Proposition 2.3. *Let* G *be a connected graph of size* $m \geqslant 3$. *Then* $\gamma'_{dR}(G) = 5$ *if and only if* $G \in \mathcal{G}$.

Proof. Assume that $\gamma'_{dR}(G) = 5$. Let $f = (E_0^f, E_2^f, E_3^f)$ be a $\gamma'_{dR}(G)$-function such that $|E_3^f|$ is maximum. Assume that $|E_3^f| = 1$ then without loss of generality, assume that $|E_2^f| = 1$. Let $E_3^f = \{uv\}$ and $E_2^f = \{xy\}$. Then, each edge with exception of xy is incident with u or v. Since G is connected and f is a $\gamma'_{dR}(G)$-function, we may assume without loss of generality, that $N(x) \cap \{u, v\} \neq \emptyset$. Assume $N(y) \cap \{u, v\} \neq \emptyset$. Then clearly $\Gamma'_{dR}(G - xy) = 3$ and $f|_{G-xy}$ is a $\gamma'_{dR}(G - xy)$-function. By proposition 2.1, $G - xy \in \mathcal{T}$. Consequently, $G \in \mathcal{G}$. Thus assume that $N(y) \cap \{u, v\} = \emptyset$. Then clearly $\gamma'_{dR}(G - y) = 3$ and $f|_{G-y}$ is a $\gamma'_{dR}(G - xy)$-function. By proposition 2.1, $G - y \in \mathcal{T}$, then clearly $G \in \mathcal{G}$. The converse is obvious. □

Theorem 2.4. *For any graph* G *of order* $n \geqslant 4$, $\gamma'_{dR}(G) \leqslant 2n - 4$. *Equality holds if and only if* $G = \{C_4, K_4, C_5, K_5, K_{2,3}, A_1, A_2, A_3, A_4, A_5, A_6\}$.

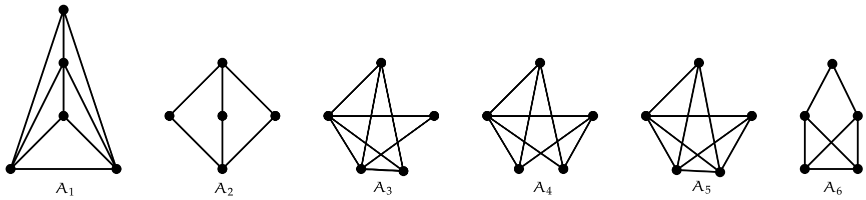

Figure 2:

Proof. Let $n = 4$ and M be a maximum matching in G. We assigning 2 to the edges of M and 0 to each other edge produces a EDRDF, implying that $\gamma'_{dR}(G) \leqslant 4 = 2n-4$. Now assume $n \geqslant 5$. Let M be a maximum matching in G. Clearly, $|M| \leqslant \frac{n}{2}$. If $|M| < \lfloor\frac{n}{2}\rfloor$ then 3 to the each of M and 0 to each other edge produces a EDRDF, implying that $\gamma'_{dR}(G) \leqslant 3\lfloor\frac{n}{2}\rfloor < 2n-4$. Thus, assume that $|M| = \frac{n}{2}$. Let $\{e_1, e_2, ..., e_{\lfloor\frac{n}{2}\rfloor}\}$ be a maximum matching of G. If n is even then assigning 2 to e_i for $i \in \{1, 2, ..., \frac{n}{2}$ and 0 to each other edge produces a EDRDF for G, thus $\gamma'_{dR}(G) \leqslant 2(\frac{n}{2}) < 2n-4$. If n is odd then assigning 2 to e_i for $i \in \{1, 2, ..., \lfloor\frac{n}{2}\rfloor + 1\}$ and 0 to each other edge produces a EDRDF for G, thus $\gamma'_{dR}(G) \leqslant 2\lfloor\frac{n}{2}\rfloor + 2 \leqslant 2n-4$. Now, assume that equality holds. By the above argument $n = 4$ it is not hard to see that $G = C_4, K_4$ and $|M| = \lfloor\frac{n}{2}\rfloor$ for n odd, it is not hard to see that $n = 5$ therefore $G \in C_5, K_5, K_{2,3}, A_1, A_2, ..., A_6$.
The converse is obvious. □

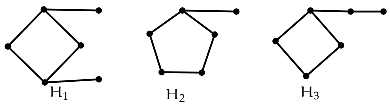

Figure 3:

Theorem 2.5. *For any triangle-free graph* G *of order* $n \leqslant 4$, $\gamma'_{dR}(G) = 2n-6$ *if and only if* $G \in P_6, C_6, C_7, H_1, H_2, H_3$.

Proof. Let G be a graph of order $n \geqslant 4$ and $\gamma'_{dR}(G) = 2n-6$. Let $C_g = (x_1x_2...x_{g(G)})$ be a shortest cycle in G. Assume that $g(G) \geqslant 8$. Let I be the set of isolated vertices of $G - C_g$ and J be the set of vertices of all K_2- components of $G - C_g$. We assign the values of a $\gamma'_{dR}(C_g)$-function to the edges of $H = G - C_g - (I \cup J)$ and 2 to each incident to any vertex of $I \cup J$. Clearly, there is precisely one edge incident to any vertex of I. Furthermore, by theorem 2.4, $\gamma'_{dR}(H) \leqslant 2(n - g(G) - |I| - |J|) - 4$ and therefore $\gamma'_{dR}(G) \leqslant 3\lceil\frac{g}{2}\rceil + |J| + 2(n - g(G) - |I| - |J|) - 4 \leqslant 2n + 3\lceil\frac{g}{2}\rceil - 2g - 4$. If G is even then, we obtain a EDRDF for G of weight less than $2n-8$, a contradiction. If G is odd then, we obtain a EDRDF for G of weight less than $2n-7$, a contradiction. We conclude that $g(G) \leqslant 7$. We continue with the following cases.
Case 1. $g(G) = 7$.

Let I be the set of isolated vertices of $G - C_7$ and J be the set of vertices of all K_2- components of $G - C_7$. We assign the values of a $\gamma'_{dR}(C_7)$-function to the edges of $H = G - C_7 - (I \cup J)$ and 2 to each incident to any vertex of $I \cup J$. Clearly, there is precisely one edge incident to any vertex of I. Furthermore, by theorem 2.4, $\gamma'_{dR}(H) \leqslant 2(n-7-|I|-|J|) - 4$. If $H \neq \emptyset$ then $\gamma'_{dR}(G) \leqslant 2\lceil\frac{7}{2}\rceil + |J| + 2(n-7-|I|-|J|) - 4 \leqslant 2n - 10 - |I| - |J|$, a contradiction. Thus $H \neq$. if $|I| \neq \emptyset$ and $|J| = \emptyset$ then $2n-6 \leqslant 8 + |I|$, it can be easily seen $|I| \leqslant 0$. Similarity, $|J| \leqslant 0$. Thus assume that $H = |I| = |J| = \emptyset$. Consequently, $G = C_7$.
Case 2. $g(G) = 6$.

Let I be the set of isolated vertices of $G - C_6$ and J be the set of vertices of all K_2- components of $G - C_6$. We assign the values of a $\gamma'_{dR}(C_6)$-function to the edges of $H = G - C_6 - (I \cup J)$ and 2 to each incident to any vertex of $I \cup J$. Clearly, there is precisely one edge incident to any vertex of I. Furthermore, by theorem 2.4, $\gamma'_{dR}(H) \leqslant 2(n-6-|I|-|J|) - 4$. If $H \neq \emptyset$ then $\gamma'_{dR}(G) \leqslant 2\lceil\frac{6}{2}\rceil + |J| + 2(n-6-|I|-|J|) - 4 \leqslant 2n - 10 - |I| - |J|$, a contradiction. Thus $H \neq$. if $|I| \neq \emptyset$ and $|J| = \emptyset$ then $2n-6 \leqslant 6+|I|$, it can be easily seen $|I| \leqslant 0$. Similarity, $|J| \leqslant 0$. Thus assume that $H = |I| = |J| = \emptyset$. Consequently, $G = C_6$.

Case 3. $g(G) = 5$.

Let I be the set of isolated vertices of $G - C_5$ and J be the set of vertices of all K_2- components of $G - C_5$. We assign the values of a $\gamma'_{dR}(C_5)$-function to the edges of $H = G - C_5 - (I \cup J)$ and 2 to each incident to any vertex of $I \cup J$. Clearly, there is precisely one edge incident to any vertex of I. Furthermore, by theorem 2.4, $\gamma'_{dR}(H) \leqslant 2(n-5-|I|-|J|)-4$. If $H \neq \emptyset$ then $\gamma'_{dR}(G) \leqslant 2\lceil\frac{5}{2}\rceil+|J|+2(n-5-|I|-|J|)-4 \leqslant 2n-8-|I|-|J|$, a contradiction. Thus $H \neq$. if $|I| \neq \emptyset$ and $|J| = \emptyset$ then $2n-6 \leqslant 8+|I|$, it can be easily seen $|I| \leqslant 2$.Let $|I| = 2$. Without less of generality assume that $y_1 \in N(x_1)-\{x_2,x_5\}$ and $y_2 \in N(x_3)-\{x_2,x_4\}$. We assign 3 to x_1x_2, x_3x_4 and 0 to $x_1y_1, x_1x_5, x_2x_3, x_3y_2, x_4x_5$, then we obtain a EDRDF for G of weight less than $2n-6$, a contradiction. Assume that $|I| = 1$ and $y_1 \in N(x_2)-\{x_1,x_3\}$. We assign 3 to x_1x_2, x_4x_5 and 0 to each edge incident to x_1x_2, x_4x_5, then we obtain a EDRDF for G of weight $2n-6$, therefore $G = H_1$. Similarity, $|J| \leqslant 2$. Without less of generality assume that $y_1 \in N(x_1)-\{x_2,x_5\}$ and $y_2 \in N(y_1)-\{x_1\}$. We assign 3 to x_1y_1, x_3x_4 and 0 to $x_1x_2, x_1x_5, x_2x_3, x_3y_2, x_4x_5$, then we obtain a EDRDF for G of weight less than $2n-6$, a contradiction. Thus assume that $H = |I| = |J| = \emptyset$. Consequently, $G = C_5$ a contradiction since $\gamma'_{dR}(C_5) = 6 \neq 2n-6$.

Case 4. $g(G) = 4$.

Let I be the set of isolated vertices of $G - C_4$ and J be the set of vertices of all K_2- components of $G - C_4$. We assign the values of a $\gamma'_{dR}(C_4)$-function to the edges of $H = G - C_4 - (I \cup J)$ and 2 to each incident to any vertex of $I \cup J$. Clearly, there is precisely one edge incident to any vertex of I. Furthermore, by theorem 2.4, $\gamma'_{dR}(H) \leqslant 2(n-4-|I|-|J|)-4$. If $H \neq \emptyset$ then $\gamma'_{dR}(G) \leqslant 4+|J|+2(n-4-|I|-|J|)-4 \leqslant 2n-8-|I|-|J|$, a contradiction. Thus $H \neq$. if $|I| \neq \emptyset$ and $|J| = \emptyset$ then $2n-6 \leqslant 8+|I|$, it can be easily seen $|I| \leqslant 2$.Let $|I| = 2$. Without less of generality assume that $y_1 \in N(x_1)-\{x_2,x_4\}$ and $y_2 \in N(x_3)-\{x_2,x_4\}$. We assign 3 to x_1y_1, x_3y_2 and 0 to $x1x_2, x_2x_3, x_3x_4, x_1x_4$, then we obtain a EDRDF for G of weight than $2n-6$. Therefore $G = H_2$. Assume that $|I| = 1$ and $y_1 \in N(x_1)-\{x_1,x_4\}$. We assign 2 to x_1x_2, x_3x_4, 1 to x_1y_1 and 0 to x_1x_3, x_2x_4, then we obtain a EDRDF for G of weight less than $2n-6$, a contradiction. Similarity, $|J| \leqslant 2$. Without less of generality assume that $y_1 \in N(x_1)-\{x_2,x_4\}$ and $y_2 \in N(y_1)-\{x_1\}$. We assign 2 to x_1x_4, x_2x_3, y_1y_2 and 0 to x_1y_1, x_1x_2, x_3x_4, then we obtain a EDRDF for G of weight $2n-6$, thus $G = H_3$. Thus assume that $H = |I| = |J| = \emptyset$. Consequently, $G = C_4$ a contradiction since $\gamma'_{dR}(C_4) = 4 \neq 2n-6$.

Case 5. $g(G) = 0$.

Thus, $G = T$ is a tree . If $diam(T) = 2$, then T is a star with at least four vertices. So $\gamma'_{dR}(G) = 3$, a contradiction. If $diam(T) = 3$, then T is a double-star with at least four vertices. So $\gamma'_{dR}(G) = 3$, a contradiction. Let P be a diametrical path. If $diam(T) \geqslant 7$ then it can be easily seen that T is EDRDF of weight less than $2n-6$, a contradiction. Thus, $4 \leqslant diam(T) \leqslant 6$. Let $\Delta(T) \geqslant 3$. Without less of generality assume that $y_1 \in N(X_4)-\{x_3,x_5\}$. We assign 3 to x_3x_4, 0 to x_2x_3, x_4y_1, x_4x_5 and 2 to each other edge produces a EDRDF for T of weight less than $2n-6$, a contradiction. Thus $\Delta(T) = 2$. If $T = P_4$ then $\gamma'_{dR}(P_4) = 4 \neq 2n-6$ and if $T = P_5$ then $\gamma'_{dR}(P_5) = 6 \neq 2n-6$. Consequently, $T = P_6$.
The converse is obvious.

□

Proposition 2.6. *Let* G *be a connected graph of size* m. *Then* $\gamma'_{dR}(G) = 2m-3$ *if and only if*

$$G = C_3, K_{1,3}, P_4, P_5$$

.

Proof. Let G be a connected graph of size m with $\gamma'_{dR}(G) = 2m-3$. By theorem **??**, $m \leqslant 4$. Thus, $diam(G) \leqslant 4$ and $g(G) \leqslant 4$. If $g(G) = 4$ then the assumption $m = 4$ implies that $G = C_4$, a contradiction since $\gamma'_{dR}(C_4) = 4$.

Assume that $g(G) = 3$. Let $C : (x_1,x_2,x_3)$ be a cycle in G. If $deg(x_1) \geqslant 3$ and $y_1 \in N(x_1)-\{x_2,x_3\}$ then assigning 3 to x_1x_2, 0 to x_1y_1, x_1x_3, x_2x_3 and 2 to each other edge produces a EDRDF for G of weight less

than $2m-3$, a contradiction. Thus, $deg(x_1) = 2$ and similarly $deg(x_2) = deg(x_3) = 2$. Consequently $G = C_3$.
Next, assume that $g(G) = 0$. Thus, $G = T$ is a tree. Suppose that T has a vertex v of degree at least 4 and $\{w_1, w_2, w_3, w_4\} \subseteq N(v)$. Then $f = (\{vw_1, vw_2, vw_3\}, E(G) - \{vw_1, vw_2, vw_3, vw_4\}, vw_4)$ is a EDRDF with $w(f) < 2m-3$, a contradiction. Thus, $\Delta(G) \leqslant 3$. If $diam(T) = 2$, then G is star and one can a easily check that $G = K_{1,3}$.
Assume that $diam(G) = 3$. Then G is a double star. If $\Delta(G) > 2$, then $f = (E(G) - \{uv\}, \emptyset, \{uv\})$, where uv is the central edge of G, is a EDRDF with $w(f) < 2m-3$, a contradiction. Thus $\Delta(G) = 2$. Consequently, $G = P_4$. It remains to assume that $diam(G) = 4$. Let $x_0x_1x_2x_3x_4$ be a diametrical path in G. If $deg(x_2) > 2$ or $deg(x_3) > 2$, then we assign 3 to x_2x_3, 0 to any edge incident with x_2x_3 and 2 to any other edge of G to obtain a EDRDF with $w(f) < 2m-3$, a contradiction. Thus, $deg(x_2) = deg(x_3) = 2$ and by symmetry, $deg(x_1) = 2$. Consequently, $G = P_5$. □

Let H be a graph obtained from C_5 by adding a leaf to one vertex of C_5. Let T_1 be a tree obtained from $P_7 : x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7$ by adding a leaf to vertex x_4 of P_7, T_2 be a tree obtained $P_6 : x_1 - x_2 - x_3x_4 - x_5 - x_6$ by adding a leaf to vertex x_3 of P_6, T_3 be a tree obtained from P_6 by adding a leaf to one support vertex of P_6, T_4 be a tree obtained from T_2 by subdividing the pendant edge incident to a vertex of degree three and T_5 be a tree obtained from P_5 by adding a leaf to a each support vertex of P_5.

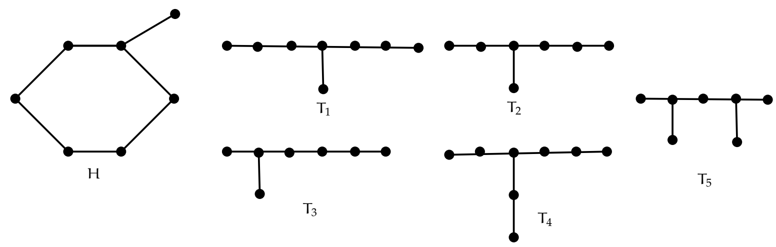

Figure 4:

Proposition 2.7. *Let G be a connected graph of size m. Then $\gamma'_{dR}(G) = 2m-6$ if and only if*

$$G \in \{C_6, C_7, H, P_7, T_1, T_2, T_3, T_4, T_5\}$$

.

Proof. Let G be a connected graph of size m with $\gamma'_{dR}(G) = 2m-6$. By theorem **??**, $m \leqslant 8$. Thus, $diam(g) \leqslant 8$ and $g(G) \leqslant 8$. Let $g(G) > 0$ and $C = (x_1, x_2, ..., x_g)$ be a cycle in G. We consider the following cases:
Case 1. $g(G) = 8$.

If $deg(x_1) > 2$ then, we assign 3 to x_1x_2 and 2 to x_3x_4, x_5x_6, x_7x_8, 0 to each other edge of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) = 2$ and similarly $deg(x_i) = 2$ for $i = 2, 3, ..., 8$. Consequently, $G = C_8$, a contradiction since $\gamma'_{dR}(C_8) = 8 \neq 2m-6$.
Case 2. $g(G) = 7$.

If $deg(x_1) > 2$ then, we assign 3 to x_1x_2 and x_4x_5, 0 to each edge incident with x_1, x_2, x_4or x_5 and 2 to each other edge of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) = 2$ and similarly $deg(x_i) = 2$ for $i = 2, 3, ..., 7$. Consequently, $G = C_7$.
Case 3. $g(G) = 6$.

If $deg(x_1) > 2$ then, we assign 3 to x_1x_2 and 2 to x_3x_4, x_5x_6, 0 to each other edge of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) = 2$ and similarly $deg(x_i) = 2$ for $i = 2, 3, ..., 6$. Consequently, $G = C_6$.

Case 4. $g(G) = 5$.

If $deg(x_1) > 3$ then, we assign 3 to x_1x_2 and x_4x_5, 0 to each edge incident with x_1, x_2, x_4or x_5 of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) \leqslant 3$. If $deg(x_1) = 3$ then We assign and similarly $deg(x_i) \leqslant 3$ for $i = 2,3,4,5$. With a similar argument, we observe that at least one vertex among $\{x_1, x_2, ..., x_5\}$ has degree three. Consequently, $G \in H$. Thus $deg(x_1) = 2$ and similarly $deg(x_2) = deg(x_3) = deg(x_4) = 2$, consequently $G = C_5$, a contradiction since $\gamma'_{dR}(C_5) = 6 \neq 2m-6$.
Case 5. $g(G) = 4$.

If $deg(x_1) > 2$ then, we assign 3 to x_1x_2 and 2 to x_3x_4, 0 to each other edge of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) = 2$ and similarly $deg(x_2) = deg(x_3) = deg(x_4) = 2$. Consequently $G = C_4$ a contradiction since $\gamma'_{dR}(C_4) = 4 \neq 2m-6$.
Case 6. $g(G) = 3$.

If $deg(x_1) > 2$ then, we assign 3 to x_1x_2 and 0 to each other edge of G to obtain a EDRDF for G of weight less than $2m-6$, a contradiction. Thus, $deg(x_1) = deg(x_2) = deg(x_3) = 2$, Consequently $G = C_3$, a contradiction since $\gamma'_{dR}(C_3) = 3 \neq 2m-6$.
Case 7. $g(G) = 0$.

Thus, $G = T$ is a tree. It can be easily seen that $4 \leqslant diam(T) \leqslant 6$. Let $y_1, y_2, ..., y_{diam(T)+1}$ be a diametrical path. Assume that $diam(T) = 6$. It is not hard no pair vertices of degree three. Consequently, $T \in \{T_1, P_7\}$.
Next, assume that $diam(T) = 5$. It is not hard to see that each vertex of T has degree at most three and there is no pair vertices of degree three. Consequently, $T \in \{T_2, T_3, T_4\}$.
Next, assume that $diam(T) = 4$. It is not hard to see that each vertex of T has degree at most three. Consequently, $T \in T_5$. □

A graph G is called a edge double Roman graph when $\gamma'_{dR}(G) = 3\gamma'(G)$. In other words, one can find a minimum edge dominating function for G using only labels 2.

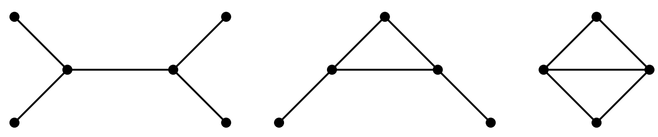

Figure 5:

Proposition 2.8. *Let G be a graph of size m. If $\Delta(G) \leqslant 3$, then $\frac{3m}{5} \leqslant \gamma'_{dR}(G)$. The equality holds if and only if G is edge double Roman graph and G is decomposed in to some copies of B_1, B_2 and B_3.*

Proof. Suppose that $f = (E_0, E_2, E_3)$ is a γ'_{dR}-function for G. Since $\Delta(G) \leqslant 3$, $|N_G[e]| \leqslant 5$, for every $e \in E(G)$ and the equality holds if and only if both vertices of e are of degree 3. therefore

$$m - |E_2| \leqslant \textstyle\sum_{e \in E_3} |N(e)| \leqslant 5|E_3|,$$

and consequently

$$\tfrac{3m+7|E_2|}{5} \leqslant 3|E_3| + 2|E_2| = \gamma'_{dR}(G).$$

If the equality holds, then $E_2 = \emptyset$, $|N_G[e]| = 5$ for every $e \in E_3$ and $N[e] \bigcap N[e'] = \emptyset$, for every two distinct edges in E_3. So G is a edge double Roman graph and also by the above argument $G[N[e]]$ is a copy of B_1, B_2or B_3 and then G is decomposed into some copies of B_1, B_2 and B_3. □

A *removable triple* of a graph G is a triple (S, M_2, M_1), where S is a nonempty subset of $V(G)$ and M_2, M_1 are disjoint matching in $G[S]$ such that every edge $e \in E(G) - M_1$ incident to a vertex in S is adjacent to some edges in M_2. We define the ratio $\rho(S, M_2, M_1)$ of a removable triple (S, M_2, M_1) to be $\frac{3|M_2|+2|M_1|}{|S|}$.

Proposition 2.9. *If a graph G has a removable triple (S, M_2, M_1) with $\rho(S, M_2, M_1) \leqslant \alpha$, then $\gamma'_{dR}(G) \leqslant \gamma'_{dR}(G - S) + \alpha|S|$.*

Proof. Let $G' = G - S$ and let f' be an edge double Roman dominating function of G' with the minimum weight. Define a function $f : E(G) \longrightarrow \{0, 1, 2, 3\}$ by setting

$$f(e) = \begin{cases} f'(e) & \text{if} \quad e \in E(G') \\ 2 & \text{if} \quad e \in M_2 \\ 3 & \text{if} \quad e \in M_1 \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

suppose e is an edge with $f(e) = 0$. If $e \in E(G')$, then e is adjacent to an edge $e' \in E(G')$, with $f(e') = f'(e') = 3$. If $e \in\notin E(G'')$, then e is incident to some vertex in S and so by the definition of a removable triple e is adjacent to some edge $e' \in M_2$ with $f(e') = 3$. Hence, f is an edge double Roman dominating function of G and so $\gamma'_{dR}(G) \leqslant \gamma'_{dR}(G') + 3|M_2| + 2|M_1| \leqslant \gamma'_{dR}(G - S) + \alpha|S|$. □

3. Edge double Roman domination and edge Roman domination

By proposition **??**, for any edge double Roman dominating function g', there exists a edge double Roman dominating g of no greater weight than g' for which $E_1 = \emptyset$. Henceforth, without loss of generality, in determining the value $\gamma'_{dR}(G)$ for any graph G, we can assume that $E_1 = \emptyset$ for all edge double Roman dominating functions under consideration.

Proposition 3.1. *Let G be a graph and $f = (E_0, E_1, E_2)$ a $\gamma'_R(G)$-function of G. Then $\gamma'_{dR}(G) \leqslant 2|E_2| + 3|E_3|$.*

Proof. Let G be a graph and $f = (E_0, E_1, E_2)$ be a γ'_R-function of G. We define a function $g = (E'_0, E'_2, E'_3)$ as follows: $E'_0 = E_0, E'_2 = E_1$, and $E'_3 = E_2$. Note that under g, every edge assigned a 0 has a neighbor assigned 3, and no edge is assigned 1. Hence, g is edge double Roman dominating function. Thus, $\gamma'_{dR}(G) \leqslant 2|E'_2| + 3|E'_3| = 2|E_1| + 3|E_2|$, as desired. □

Clearly, the bound of proposition 3.1is sharp, as can be seen with the family stars $G = K_{1,n-1}$, where $\gamma'_R(G) = 2$ and $\gamma'_{dR}(G) = 3$. We also note that strict inequality in the bound can be achieved. Consider the subdivided star $G = K^*_{1,k}$, formed by subdividing each edge of the star $K_{1,k}$ with center u and $V(K_{1,k}) = \{u, v_i, w_j : 1 \leqslant i, j \leqslant k\}$ and $E(K_{1,k}) = \{uv_i, v_iw_j : 1 \leqslant i, j \leqslant k\}$, for $k \geqslant 3$, exactly once. We note that $\gamma'_R(G) = k + 1$ and $\gamma'_{dR}(G) = 2k + 1$. To see this, assign to each edgesv_iw_j except v_1w_1, 2 to the uv_1, and 0 otherwise for a edge Roman dominating function $f = (E_0, E_1, E_2)$; and assign 2 to edges uv_1, v_iw_j for $2 \leqslant i, j \leqslant k$, and 0 otherwise for a edge double Roman dominating function. It is simple to check that these functions are optimal. Hence, $|E_1| = k$ and $|E_2| = 1$, and so, $2k + 1 = \gamma'_{dR}(G) < 2|E_1| + 3|E_2| = 2k + 3$.

Corollary 3.2. *For any graph G, $\gamma'_{dR}(G) \leqslant 2\gamma'_R(G)$, with equality if and only if $G = mK_2$.*

Proof. Among all γ'_R-functions of G, let $f = (E_0, E_1, E_2)$ be one that minimizes the number of edges in E_1. Since $\gamma'_R(G) = |E_1| + 2|E_2|$, by proposition 3.1, we have that $\gamma'_{dR}(G) \leqslant 2|E_2| + 3|E_2| = \gamma'_R(G) + |E_1| + |E_2| \leqslant 2\gamma'_R(G)$.
If $\gamma'_{dR}(G) = 2\gamma'_R(G) = 2|E_1| + 4|E_2|$, then since $\gamma'_{dR}(G) \leqslant 2|E_1| + 3|E_2|$, we must have that $E_2 = \emptyset$. Hence, $E_0 = \emptyset$ must hold, and so $E = E_1$. Since $|E_1|$ is minimized under f, we deduce that no two edges in G are adjacent, for otherwise, if e and e' are adjacent, then the function f' which assigns a 0 to e, a 2 to e', and a 1 to every other vertex is a γ_R-function of G having a smaller number of edges assigned 1 than f does. Thus, $G = mK_2$. □

From this, the next corollary is immediate.

Corollary 3.3. *If* G *is a nontrivial, connected graph, and* $f = (E_0, E_1, E_2)$ *is a* γ'_R*-function of* G *that maximizes the number of edges in* E_2*, then* $\gamma'_{dR}(G) \leqslant 2\gamma'_R(G) - |E_2|$.

By corollary 3.3, we see that for a connected graph G, edge double Roman domination does in fact provide edge double Roman domination does in fact provide edge double the protection with strictly less than edge double the cost of a edge Roman dominating function. For example, let G be a nontrivial graph with $\gamma'(G) = 1$. Then $\gamma'_R(G) = 2$ and $\gamma'_{dR}(G) = 3$. On the other hand, using the example of the subdivided star $G^* = K^*_{1,k}$, with $k \geqslant 3$, we see that the $\gamma'_{dR}(G)$ approaches $2\gamma'_R(G)$ for some graphs. Recall that $\gamma'_R(G^*) = k+1$ and $\gamma'_{dR}(G^*) = 2k+1$, and thus, the ratio of $\gamma'_{dR}(G^*)$ to $\gamma'_R(G^*)$ is $\frac{2k+1}{k+1}$ and approaches 2 as k approaches infinity.
Next we see that the edge Roman domination number is strictly smaller than the edge double Roman domination number.

Proposition 3.4. *For every graph* G, $\gamma'_R(G) < \gamma'_{dR}(G)$.

Proof. Let $f = (E_0, E_2, E_3)$ be any γ'_{dR}-function of G, where $E_1 = \emptyset$(by proposition **??** such a function exists). If $E_3 \neq \emptyset$, then every vertex in E_3 can be reassigned the value 2 and the resulting function will be a edge Roman dominating function, that is, $\gamma'_R(G) < \gamma'_{dR}(G)$.
Assume then that $E_3 = \emptyset$. Since $E_2 \cup E_3$ dominates G, it follows that $E_2 \neq \emptyset$. Thus, all vertices are assigned either the value 0 or the value 2, and all edges in E_0 must have at least two neighbors in E_2. In this case one vertex in E_2 can be reassigned the value 1 and the resulting function will be a edge Roman dominating function, that is, $\gamma'_R(G) < \gamma'_{dR}(G)$. □

Corollary 3.5. *If* $f = (E_0, E_2, E_3)$ *is any* γ'_{dR}*-function of a graph* G*, then*

$$\gamma'_R(G) \leqslant 2(|E_2| + |E_3|) = \gamma'_{dR}(G) - |E_3|.$$

Corollary 3.6. *For any nontrivial connected graph* G, $\gamma'_R(G) < \gamma'_{dR}(G) < 2\gamma'_R(G)$.

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